

Initial-Value Problem for Inhomogeneous Condensate: Gaussian Approximation and Beyond

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ABSTRACT

Using many-body theory we develop a set of formally exact kinetic equations for inhomogeneous condensate and one-body observables. The method is illustrated for ϕ^4 field theory in 1+1 dimensions. These equations, when computed with the help of time-dependent projection technique, lead to a systematic mean-field expansion. The lowest and the higher order terms correspond to, respectively, the gaussian approximation and the dynamical correlation effect.

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I. Introduction

The dynamical evolution of inhomogeneous field configurations is an important problem and a common theme in cosmology, high energy and condensate matter physics. In cosmology inhomogeneous field configurations appear when topological objects such as textures or cosmic strings involving inhomogeneous field configurations are considered. Their relaxational dynamics is thought to have a bearing on the spectrum of fluctuations in the cosmic microwave background radiation [1]. In the ultrahigh energy heavy-ion collisions ($\sqrt{s} \geq 200$ GeV/nucleon) a large energy density (few GeV/fm³) is deposited in the collision region corresponding to temperature above the critical value for chiral symmetry restoration. In this situation, it is possible within a volume of few fm³ an inhomogeneous condensate is formed. When these regions cool down this field relax toward the equilibrium situation which might be misaligned with the vacuum state [2]. On the other hand, the recent success of the experimental observation of Bose-Einstein condensation for systems of spin polarized magnetically trapped alkali atoms at ultra-low temperature is stimulating development of theory for the evolution of nonuniform condensates [3].

A microscopic description of this off-equilibrium process requires a nonperturbative treatment of inhomogeneous condensates and in the field-theoretical context, this has been implemented through the use of a Gaussian ansatz for the wavefunctional in the framework of a time-dependent variational principle [4]. Actually, the Gaussian ansatz, having the form of an exponential of quadratic form in the field operators, implies the many-point functions, can be in fact factored in terms of two-point functions. The dynamics of the reduced two-point density becomes then itself isoentropic, as a result of irreducible higher-order correlation effects being neglected.

The purpose of the present paper is twofold. The first one is to reevaluate and improve the gaussian approximation. We follow a time-dependent approach developed earlier in the context of nuclear many-body dynamics [5]. This method allows for a formulation of a mean-field expansion for the dynamics of the two-point correlation function from which one recovers the results of the gaussian mean-field approximation in lowest order. Beyond this, we are able to explicitly include higher dynamical correlation effects. The second purpose is to extend the former results in the context of spatial uniformity to the inhomogeneous field configurations [6]. In this case the spatial dependence of the field operator is expanded in the general natural orbitals. These orbitals can be given in terms of an expansion of convenient basis which will also evolve in time according to additional dynamical equations. Although the procedure is quite general, we will apply our method in the context of a single scalar

field in 1+1 dimensions. This will illustrate all the relevant points of the approach and cut down inessential technical complications.

II. A Simple Example in Quantum Mechanics

This section will illustrate the scheme of approximation in the context of ϕ^4 in $0 + 1$ dimension [7]. Although this theory is finite, we introduce nevertheless a counterterm for the further use. Therefore the hamiltonian reads as

$$\hat{H} = \frac{1}{2}\hat{\pi}^2 + \frac{m^2}{2}\hat{\phi}^2 + \frac{g}{24}\hat{\phi}^4 - \frac{g}{8m}\hat{\phi}^2 \quad (2.1)$$

where m is the renormalized mass. In the usual quantum mechanics (QM) language, one speaks of a particle in the quartic potential with ϕ and π being its position and momentum operator and satisfy the usual commutation relation, $[\phi, \pi] = i$. In the quantum field theory (QFT) language, however, one imagines that this field lives in *one* point and the particles have quartic self-interaction [8].

A natural choice of the subsystem in the context of many-body problem is the observables associated to the one-body density. The exact microscopic description of the time evolution of the one-body density of a many-body system can be formally given in terms of a sum of two parts: the usual gaussian contribution (also known as time-dependent hartree-bogoliubov approximation) plus additional dynamics arise from the time evolution of quantum correlations in the entire system. The physical origin of the later contribution lies in the complicated dynamical evolution of quantum correlation in the entire system and their consequences in the change in the coherence properties of each system.

II-a. Gaussian Variables

In order to derive the dynamical equations for the one-body observables we focus on the operators which are either linear or bilinear forms of creation and annihilation, henceforth referred to as gaussian observables. We begin therefore write the Heisenberg field operators ϕ and π as

$$\phi(t) = \frac{1}{\sqrt{2\mu}}[a^\dagger(t) + a(t)] \quad \pi(t) = i\sqrt{\frac{\mu}{2}}[a^\dagger(t) - a(t)] \quad (2.2)$$

where a, a^\dagger are the usual annihilation and creation operators satisfying the boson commutation relations: $[a, a^\dagger] = 1$; the parameter μ will be fixed later in a convenient way. The state of the entire system is given in terms of the matrix density F in the Heisenberg picture. It is Hermitean, time independent and has a unit trace.

The first variable of interest is given as mean value of annihilation,

$$A_t = \text{Tra}(t)F, \quad (2.3)$$

and its complex conjugate. From this one can define the shifted boson operator

$$b = a - A. \quad (2.4)$$

Next, we consider the possible mean values of the bilinear forms of the shifted operators, which can be combined in an extended one boson plus pairing density,

$$R = \begin{pmatrix} \langle b^\dagger(t)b(t) \rangle & \langle b(t)b(t) \rangle \\ \langle b^\dagger(t)b^\dagger(t) \rangle & \langle b(t)b^\dagger(t) \rangle \end{pmatrix} = \begin{pmatrix} \Lambda & \Pi \\ \Pi^* & 1 + \Lambda^* \end{pmatrix} = R^\dagger \quad (2.5)$$

The quantities Λ and Π together with the mean-value of field, A , describe the one-body observables of the system.

To deal with the pairing density Π we proceed, as usual, by defining the Bogoliubov quasi-particle operator as [9]

$$d(t) = x_t^* b(t) + y_t^* b^\dagger(t) \quad d^\dagger(t) = x_t b^\dagger(t) + y_t b(t) \quad (2.6)$$

and require that $\langle dd \rangle = \langle d^\dagger d^\dagger \rangle = 0$. A sistematic way to determine the coefficients of the Bogoliubov transformation, x_t and y_t , is to solve the follwoing secular problem:

$$G R X = X G N, \quad (2.7)$$

where

$$G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} x_t & y_t^* \\ y_t & x^* \end{pmatrix}, \quad N = \begin{pmatrix} \nu_t & 0 \\ 0 & 1 + \nu_t \end{pmatrix}. \quad (2.8)$$

The eigenvalues $\nu_t = \text{Tr} d^\dagger(t) d(t) F$ stand for occupation numbers of quasi-particles. Since the Bogoliubov transformation is canonica one can verify that X_t satisfies the orthogonality and completeness relation, i.e.,

$$X^\dagger G X = X G X^\dagger = G \quad (2.9)$$

In terms of coefficients of transformation (2.9) reads as

$$|x_t|^2 - |y_t|^2 = 1 \quad (2.10)$$

Thus, the pairing densities $\langle d(t)d(t) \rangle$ and $\langle d^\dagger(t)d^\dagger(t) \rangle$ can be obtained from x_t , y_t and ν_t . The next step is to get equations of motion for these quantities.

II-b. Equation of Motion and Gaussian Approximation

The Next step isto obtain the time evolution for the variables of interest discussed in the previous subsection. We begin with the amplitude of condensate A_t defined by Eq.(2.3). Using Heisenberg equation of motion one has

$$i\dot{A}_t = \text{Tr}[a, H]F = x_t \text{Tr}[d, H]F + \text{Tr}y^*[d^\dagger, H]F \quad (2.11)$$

where we have used (2.4) and (2.6). Now, to get equations for the Bogoliubov coefficients we rewrite (2.7), using (2.9), as

$$X_t^\dagger R X_t = N. \quad (2.12)$$

Taking time derivative of this and using Eq.(2.9) we get

$$X_t^\dagger \dot{R} X_t = \dot{N} - \dot{X}_t^\dagger R X_t + X_t^\dagger R \dot{X}_t. \quad (2.13)$$

The left-hand side of this equation can be evaluated from the Heisenberg equation of motion and (2.6) and yields

$$iX_t^\dagger \dot{R} X_t = \begin{pmatrix} \text{Tr}[d^\dagger d, H]F & \text{Tr}[dd, H]F \\ \text{Tr}[d^\dagger d^\dagger]F & \text{Tr}[dd^\dagger, H]F \end{pmatrix}, \quad (2.14)$$

The right-hand side of (2.13), on the other hand, can be obtained with the help of (2.9). Equation now the result to (2.14) one gets

$$i\dot{\nu} = \text{Tr} [d^\dagger d, H]F \quad (2.15)$$

and

$$i(\dot{x}y - x\dot{y}) = \text{Tr} [d^\dagger d, H]F. \quad (2.16)$$

The equations (2.10), (2.15) and (2.16), together with the normalization condition, determine fully, in principle, the time evolution for the gaussian observables if the density matrix F is expressible in terms of the quantities themselves. However, when the hamiltoniana H involves self-interaction fields, traces of these equations will involve also many-body densities and therefore they are not closed. One emergent approximation to deal with this situation is to replace the full density F by a truncated one, F_0 , which has the form of a exponential

of bilinear in creation and annihilation operators (see, e.g., Eq.(2.22) of [6]). In terms of quasi-particle operator F_0 can be written conveniently as

$$F_0(t) = \frac{1}{1 + \nu_t} \left(\frac{\nu_t}{1 + \nu_t} \right)^{d^\dagger(t)d(t)} \quad (2.17)$$

Notice that F_0 is diagonal in the quasi-particle basis and contain no irreducible two or many quasi-particle correlations. Furthermore, it is easy to verify that F_0 given by (2.17) has a unit trace and reproduce the corresponding average of linear and bilinear field oprarors of full density, i.e.,

$$\begin{aligned} \text{Tr } aF_0 &= A = \text{Tr } aF \\ \text{Tr } d^\dagger dF_0 &= \nu = \text{Tr } d^\dagger dF \\ \text{Tr } ddF_0 &= 0 = \text{Tr } ddF \end{aligned}$$

Therefore the approximation gives a set of self-consistent equations for the one-body observables and is what we refer as the mean-field or gaussian approximation. In particular, when F_0 is written in terms of field representation, it is equivalent to the density used by Jackiw [4]. This approximation constrains the time evolution of the system to remain in a gaussian, which contains no irreducible two or many particle correlation, whereas the true evolution will, as time progresses, introduce (quasi)particle-particle correlation not describable by the gaussian-like matrix density. In terms of general discussion made in the previous subsection the limitation of the approximation shows up in the dynamical evolution of occupancy,

$$i\dot{\nu} = \text{Tr } [d^\dagger d, H]F_0 = \text{Tr } [d^\dagger d, F_0]H = 0. \quad (2.18)$$

Therefore , further improvements have to be achieved in order to describe correlation effects between different subsystems.

II-c. Projection Technique and Dynamical Correlations

The question we want to address ourselves at this point is how to express the full correlated density F of entire system in terms ingredients of subsystem (gaussian variables) in such way that one can get a set of closed equations for the one-body observable. A framework to achieve this goal was developed some time ago by Willis and Picard using time-dependent projection technique [10] in the context of master equation for coupled systems. The method was extended later by Nemes and Piza to study nuclear many-body dynamics [5]. The method consists essentially in writing the correlation informations of the full density in teerms of a memory kernel acting on the uncorrelated density F_0 .

Following their strategy we first decompose F in two parts

$$F = F_0(t) + F'(t) \quad (2.19)$$

where $F_0(t)$ is the uncorrelated part given by (2.17) and therefore $F'(t)$ is a traceless correlation part. The substitution of F by just $F_0(t)$ in the equations (2.11) and (2.15)-(2.16) gives the usual gaussian approximation as we have discussed before. The crucial step is to observe that $F_0(t)$ can be seen as a time-dependent projection of F , i.e.,

$$F_0 = \mathbb{P}(t) F \quad , \quad \mathbb{P}(t) \mathbb{P}(t) = \mathbb{P}(t) \quad . \quad (2.20)$$

For the explicit construction of $\mathbb{P}(t)$ we require, in addition to eqs. (2.20), the condition

$$i \dot{F}_0(t) = [F_0(t), H] + \mathbb{P}(t)[H, F] \quad (2.21)$$

which is the Heisenberg picture counterpart of the Schrödinger picture condition used in [11] to determine \mathbb{P} uniquely. The resulting form for $\mathbb{P}(t)$ is (see appendix A of Ref.[6] for details of the derivation)

$$\begin{aligned} \mathbb{P} \cdot = & \left\{ \left[1 - \frac{d^\dagger d - p}{1+p} \right] \text{Tr}(\cdot) + \frac{d^\dagger d - p}{p(1+p)} \text{Tr}(d^\dagger d \cdot) + \left[\frac{d}{p} \text{Tr}(d^\dagger \cdot) + \frac{d^\dagger}{1+p} \text{Tr}(d \cdot) \right] \right. \\ & \left. + \left[\frac{d d}{2p p} \text{Tr}(d^\dagger d^\dagger \cdot) + \frac{d^\dagger d^\dagger}{2(1+p)(1+p)} \text{Tr}(d d \cdot) \right] \right\} F_0 \quad . \end{aligned} \quad (2.22)$$

where the dot stands for objects on which the projector acts.

The next step is to obtain a differential equation for the correlated density $F'(t)$. This follows immediately from eqs. (2.19) and (2.21),

$$\left(i \frac{d}{dt} - \mathbb{P}(t) \mathbb{L} \right) F'(t) = \mathbb{Q}(t) \mathbb{L} F_0(t) \quad , \quad (2.23)$$

where we have introduced the operators

$$\mathbb{Q}(t) = \mathbb{L} - \mathbb{P}(t) \quad , \quad \mathbb{L} \cdot = [H, \cdot] \quad . \quad (2.24)$$

This equation has the formal solution

$$F'(t) = \mathbb{G}(t, 0) F'(0) - i \int_0^t dt' \mathbb{G}(t, t') \mathbb{Q}(t') \mathbb{L} F_0(t') \quad , \quad (2.25)$$

where $\mathbb{G}(t, t')$ is the time-ordered Green's Function

$$\mathbb{G}(t, t') = T \exp i \int_{t'}^t d\tau \mathbb{P}(\tau) L \quad . \quad (2.26)$$

What we have obtained so far is a formally exact expression relating $F'(t)$ and $F_{t'}$ (for $t' \leq t$) and the initial correlations $F'(0)$. This allows us to get a set of closed dynamical equations as traces over functional of $F_0(t')$ and initial correlations. An actual calculation is, however, hopeless because of the complicated time dependence of the Heisenberg operators present in the memory kernel of (2.25). A systematic expansion to treat the memory integral of the equation (3.8) has been discussed in Ref.[11] in the Schrödinger picture. The implementation of the corresponding expansion in the Heisenberg picture consists in approximating the time evolution of the field operators, when evaluating memory effects, by a simpler one-body generator of mean-field evolution, i.e.,

$$i\dot{d} = [d, H_0] - i\dot{A} + i(\dot{x}_t^* x_t - \dot{y}_t^* y_t)d - i(\dot{x}_t^* x_t^* - \dot{y}_t^* y_t^*)d^\dagger. \quad (2.27)$$

The last three terms account for the (explicit) time dependence of $d(t)$ on condensate and pairing effects; The hamiltonian H_0 is taken, in this approximation, as a mean-field one,

$$\begin{aligned} H_0 = & P^\dagger H + d^\dagger \text{Tr}[d, H] F' - d \text{Tr}[d^\dagger, H] F' \\ & + \frac{d^\dagger d^\dagger}{2(1+2p)} \text{Tr}[d d, H] F' - \frac{d d}{2(1+2p)} \text{Tr}[d^\dagger d^\dagger, H] F'. \end{aligned} \quad (2.28)$$

This approximation results a unitary time evolution for the Heisenberg field operator and operators in different time are related by a phase factor

$$d(t) = e^{i\varphi(t,t')} d(t'). \quad (2.29)$$

(see the third equation on page 1609 of [7]; see also (5.3) of [6] for details).

In this way, the authors of the ref.[11] devised a systematic expansion for the correlation density $F'(t)$ [see their Eq.(3.10)]. The corresponding expansion in the Heisenberg picture reads as

$$\begin{aligned} F'(t) = & \mathbb{G}(t, 0) F'(0) - i \int_0^t dt'_1 \mathbb{Q}(t_1) \mathbb{L} F_0(t_1) \\ & - \int_0^t dt_1 \left[\int_{t_1}^t dt_2 \mathbb{Q}(t_2) (\mathbb{L} - \mathbb{L}_0(t_2)) \right] \mathbb{Q}(t_1) \mathbb{L} F_0(t_1) + \dots, \end{aligned} \quad (2.30)$$

where $\mathbb{L}_0 \cdot = [H_0, \cdot]$. In what follows we restrict ourselves to initial conditions such that $F'(0) = 0$ and to the lowest approximation for $F'(t)$. Therefore, the evaluation of the

equations of motion involves traces of the type

$$\begin{array}{ccc} \text{Tr} [\hat{O}(t), H] F_0(t) & - & i \text{Tr} [\hat{O}(t), H] \int_0^t dt' \mathbb{Q}(t') [H, F_0(t')] \\ \text{mean - field} & & \text{correlation} \end{array} \quad (2.31)$$

where $\hat{O}(t)$ can be $d(t)$, $d^\dagger(t) d(t)$, $d(t) d(t)$, and operators at different times are related by equation (2.29).

We have now all the necessary ingredients for implementation of the approximation and the derivation for the equation of motion is a straightforward algebraic exercise. The results, including the numerical calculation, for ϕ_{0+1}^4 model are shown in [7]. Extensions of this method to other models of field theory were obtained recently [6]. The results demonstrate that this approach can overcome some conceptual difficulties of gaussian approximation as well as a much better description for the gaussian observables.

III. Kinetic Equations for Simple Observables of the Field

Section II reviewed some important points of our extended gaussian approximation and its implementation in the simplest context of quantum mechanics. In this and next section we will report its applications in context of inhomogeneous field configuration, which is relevant for the dynamical evolution of many body finite system. The simpler case of spatial uniformity was discussed for several field models recently. Technical difficulties in such cases reduce tremendously because of translational invariance and the gaussian variables are automatically diagonal in momentum space. In other words, the eigenfunctions of the body-density, known as natural orbitals, are independent of time and given by plane wave. For this new scenario, however, the natural orbitals are time dependent. Therefore, additional equation is needed in order to get self-consistent equations of motion.

III-a. Generalized Bogoliubov Transformation

In order to implement the idea let us first expand the Heisenberg field operator $\phi(t, x)$ and the canonical momentum $\pi(t, x)$ as

$$\phi(t, x) = \sum_k \left[f_k(x) a_k(t) + f_k^*(x) a_k^\dagger(t) \right] , \quad (3.1)$$

$$\pi(t, x) = -i \sum_k k_0 \left[f_k(x) a_k(t) - f_k^*(x) a_k^\dagger(t) \right] , \quad (3.2)$$

where $a_k(t)$, $a_k^\dagger(t)$ are boson operators satisfying the equal time commutation relation

$$[a_k(t), a_{k'}(t)] = \delta_{kk'} . \quad (3.3)$$

The $f_k(x)$ are the periodic boundary condition plane waves

$$f_k(x) = \frac{e^{ik \cdot x}}{\sqrt{2Lk_0}} , \quad (3.4)$$

L being the lenght of the periodicity box and $k_0^2 = k^2 + \mu^2$. The expansion mass parameter μ is conveniently fixed, e.g. in terms of the equilibrium solution in the mean field approximation [6].

The next step is to focus on the variables of interest, which are mean-value of linear and bilinear boson operators. The first of them is the expectation value of the field operator,

$$\langle \phi(t, x) \rangle = \text{Tr } \phi(t, x) F , \quad (3.5)$$

where F is a density matrix in the Heisenberg picture that characterizes the state of the system. In terms of the expansion (2.1), one has

$$\langle \phi(t, x) \rangle = \sum_k [f_k(x) A_k(t) + f_k^*(x) A_k^*(t)] \quad (3.6)$$

with

$$A_k(t) = \text{Tr } a_k(t) F . \quad (3.7)$$

We can now define the shifted boson operators with the help of the $A_k(t)$,

$$b_k(t) = a_k(t) - A_k(t) , \quad (3.8)$$

and include as variables of interest also the expectation value of pairs of $b_k(t)$, $b_k^\dagger(t)$ at equal times:

$$\Lambda_{kk'}(t) = \text{Tr } b_{k'}^\dagger(t) b_k(t) F , \quad (3.9)$$

$$\Pi_{kk'}(t) = \text{Tr } b_{k'}(t) b_k(t) F . \quad (3.10)$$

The hermitean matrix Λ and the symmetric matrix Π are in fact the one-boson density matrix and the pairing density for the shifted bosons respectively. The corresponding matrices for the a -boson are easily expressed in terms of Λ , Π and A_k .

The next step is to write the one-body density matrix in diagonal form and also incorporate information of the pair density in an associated set of natural orbitals by setting up an extended one-body density matrix [9],

$$R = \begin{pmatrix} \Lambda & \Pi \\ \Pi^* & 1 + \Lambda^* \end{pmatrix} = R^\dagger \quad (3.11)$$

and solving the extended version of eigenvalue problem defined by (2.7), where the matrix elements are given now the following matrices:

$$G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} U & V^* \\ V & U^* \end{pmatrix}, \quad N = \begin{pmatrix} P & 0 \\ 0 & 1 + P \end{pmatrix}. \quad (3.12)$$

Since (2.7) is a non-hermitean eigenvalue problem, it is important to consider also the adjoint equation

$$R G \tilde{X} = \tilde{X} G N, \quad (3.13)$$

from which one finds that

$$\tilde{X} = G X. \quad (3.14)$$

The adjoint vectors \tilde{X} satisfy biorthogonality relations with X which allow one to introduce the normalization condition

$$\tilde{X}^\dagger X = X^\dagger G X = G \quad (3.15)$$

and the completeness relation

$$X G X^\dagger = G. \quad (3.16)$$

Furthermore, one can use the eigenvectors of the secular problem (2.12) to construct new boson operators

$$\begin{pmatrix} d \\ d^\dagger \end{pmatrix} = X^\dagger \begin{pmatrix} b \\ b^\dagger \end{pmatrix}. \quad (3.17)$$

This equation defines in fact the general Bogoliubov transformation

$$d_\alpha(t) = \sum_k \left(U_{k\alpha}^*(t) b_k(t) + V_{k\alpha}^*(t) b_k^\dagger(t) \right). \quad (3.18)$$

It is easy to see that the secular problem (2.12) constrains the expectation value of products of $d_\alpha(t)$, $d_\alpha^\dagger(t)$ as

$$\text{Tr } d_\alpha^\dagger(t) d_\beta(t) F = P_\alpha(t) \delta_{\alpha\beta} \quad (3.19)$$

$$\text{Tr } d_\alpha(t) d_\beta(t) F = 0. \quad (3.20)$$

where P_α are elements of the eigenvalue matrix P and can be interpreted as quasiparticle occupation numbers. With the help of the eq. (2.16) one can invert the relation (2.17) as

$$\begin{pmatrix} b \\ b^\dagger \end{pmatrix} = G X G \begin{pmatrix} d \\ d^\dagger \end{pmatrix} . \quad (3.21)$$

On the other hand, one can now express the field operator in terms of the natural orbitals as

$$\phi(t, x) = \langle \phi(t, x) \rangle + \sum_\alpha [v_\alpha(t, x) d_\alpha(t) + v_\alpha^*(t, x) d_\alpha^*(t)] \quad (3.22)$$

with

$$v_\alpha(t, x) = \sum_k [f_k(x) U_{k\alpha}(t) - f_k^*(x) V_{k\alpha}(t)] . \quad (3.23)$$

Equations have been used to obtain these results. Therefore, our treatment for the general non-uniform field configuration involves expanding the field operator in the general natural orbitals, which are in turn given in terms of plane wave (or some other convenient) expansion.

III-b. Formal Equations of Motion for the Gaussian Observables

The next step is to derive the equation of motion for these simple variables (A_k , p_α , $U_{k\alpha}$ and $V_{k\alpha}$). For $A_k(t)$ one finds immediately from the Heisenberg equation of motion

$$i \dot{A}_k(t) = \text{Tr} [a_k(t), H] F , \quad (3.24)$$

where H is the field Hamiltonian. The equation of motion for the remaining quantities can be obtained by taking the time-derivatives of the eigenvalue equation (2.12), again in close analogy with (2.13)-(2.14). Using the definitions for the matrices N , X and R , this equation can be written explicitly as

$$\begin{aligned} & \begin{pmatrix} U^\dagger \dot{A} U + U^\dagger \dot{I} V + V^\dagger \dot{I}^* U + V^\dagger \dot{A}^* V & U^\dagger \dot{A} V^* + U^\dagger \dot{I} U^* + V^\dagger \dot{I} V^* + V^\dagger \dot{A}^* U^* \\ U^\dagger \dot{A}^* V + U^\dagger \dot{I} U + V^\dagger \dot{I}^* V + V^\dagger \dot{A} U & U^\dagger \dot{A} U^* + U^\dagger \dot{I}^* V^* + V^\dagger \dot{I} U^* + V^\dagger \dot{A} V^* \end{pmatrix} \\ &= \begin{pmatrix} \dot{P} + [U^\dagger \dot{U} - V^\dagger \dot{V}, P] & V^\dagger \dot{U}^* - U^\dagger \dot{V}^* + \{V^\dagger \dot{U}^* - U^\dagger \dot{V}^*, P\}_+ \\ V^\dagger \dot{U} - U^\dagger \dot{V} + \{V^\dagger \dot{U} - U^\dagger \dot{V}, P\}_+ & \dot{P} + [U^\dagger \dot{U}^* - V^\dagger \dot{V}^*, P] \end{pmatrix} , \end{aligned} \quad (3.25)$$

where $\{ \}_+$ indicates an anticommutator. Note that there are two independent matrix equations, the remaining two being their complex conjugates. The block matrix elements in the left hand side of eq. (2.27) can be rewritten using the Heisenberg equation of motion as

$$i \left[U^\dagger \dot{U} + U^\dagger \dot{V} + V^\dagger \dot{U}^* + V^\dagger \dot{V}^* \right]_{\alpha\beta} = \text{Tr} \left[d_\beta^\dagger d_\alpha, H \right] F \quad , \quad (3.26)$$

$$i \left[U^\dagger \dot{V}^* + U^\dagger \dot{U}^* + V^\dagger \dot{V} + V^\dagger \dot{U} \right]_{\alpha\beta} = \text{Tr} \left[d_\beta d_\alpha, H \right] F \quad . \quad (3.27)$$

Equating corresponding block matrix elements on the two sides of equation (2.27) yields

$$i \left\{ \dot{P} + \left[U^\dagger \dot{U} - V^\dagger \dot{V}, P \right] \right\}_{\alpha\beta} = \text{Tr} \left[d_\beta^\dagger d_\alpha, H \right] F \quad , \quad (3.28)$$

$$i \left[V^\dagger \dot{U}^* - U^\dagger \dot{V}^* + \left\{ V^\dagger \dot{U}^* - U^\dagger \dot{V}^*, P \right\}_+ \right]_{\alpha\beta} = \text{Tr} \left[d_\beta d_\alpha, H \right] F \quad . \quad (3.29)$$

Since P is a diagonal matrix, equation (3.28) splits into two equations, one for $\alpha = \beta$ and other for $\alpha \neq \beta$,

$$i \dot{p}_\alpha = \text{Tr} \left[d_\alpha^\dagger d_\alpha, H \right] F \quad , \quad (3.30)$$

$$i (p_\beta - p_\alpha) \left(U^\dagger \dot{U} - V^\dagger \dot{V} \right)_{\alpha\beta} = \text{Tr} \left[d_\beta^\dagger d_\alpha, H \right] F \quad \alpha \neq \beta \quad . \quad (3.31)$$

Moreover, equation (2.31) can be rewritten as

$$i \left(V^\dagger \dot{U}^* + U^\dagger \dot{V}^* \right)_{\alpha\beta} (1 + p_\alpha + p_\beta) = \text{Tr} \left[d_\beta d_\alpha, H \right] F \quad . \quad (3.32)$$

For the particular case of a spatially uniform system, eq. (2.33) is trivial since the plane waves are the natural orbitals.

In order to treat the dynamics of natural orbitals, which is given in terms of the awkward looking combinations $U^\dagger \dot{U} - V^\dagger \dot{V}$ and $V^\dagger \dot{U}^* - U^\dagger \dot{V}^*$ in eqs. (2.33) and (2.34), it is convenient to first define matrices h and g as

$$h = i \left(U^\dagger \dot{U} - V^\dagger \dot{V} \right) \quad , \quad (3.33)$$

$$g = i \left(V^\dagger \dot{U}^* - U^\dagger \dot{V}^* \right) \quad . \quad (3.34)$$

The next step is to find simple relations between \dot{U} , \dot{V} and h, g . The crucial point is to observe that the dynamics of the eigenvector X can be described as being generated by a dynamical matrix Ω such that

$$i \dot{X}(t) = X(t) \Omega(t) G. \quad (3.35)$$

Using (2.16) and (2.17) one can solve eq. (2.37) for $\Omega(t)$ as

$$\Omega(t) = i G X^\dagger(t) G \dot{X}(t) G = \begin{pmatrix} h & g \\ g^* & h^* \end{pmatrix}. \quad (3.36)$$

Equating the matrix blocks of the equation (2.37) one finds

$$i \dot{U} = U h + V^* g^* \quad (3.37)$$

$$i \dot{V} = V h + U^* g^*. \quad (3.38)$$

Therefore, the final equations of motion for the simple variables of the field are eqs. (3.24), (3.30), (3.37) and (3.38). The ingredients of the matrices h and g of eqs. (3.37) and (3.38) are found from (3.33) and (3.34). These equations are of course not closed equations since they still involve the full time evolution of the field operator as we have seen in previous section.

The basic idea of our approximation scheme is described in section III. The additional difficulty because of the spatial dependence of field can be handled when orbital representation is used. The crucial point here is to notice that the reduced density $F_0(t)$ in mean-field approximation can be conveniently written as

$$F_0(t) = \prod_{\alpha} \frac{1}{1 + p_{\alpha}(t)} \left[\frac{p_{\alpha}(t)}{1 + p_{\alpha}(t)} \right] d_{\alpha}^{\dagger}(t) d_{\alpha}(t). \quad (3.39)$$

From these ingredients one can construct the correlated density $F'(t)$ and finally the equations of motion for a specific model.

IV. Mean-Field and Collisional Dynamics in ϕ^4 Field Theory

Thus far we have presented a general procedure to investigate kinetics of one-body observable in the context of a scalar field, without mentioning, however, any specific field model.

In this sections we will illustrate a specific example as an application of former more formal treatment. We consider the ϕ^4 Hamiltonian

$$H = \int dx \mathcal{H} \quad (4.1)$$

$$\mathcal{H} = \frac{\pi^2}{2} + \frac{1}{2} (\partial_x \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{g}{4!} \phi^4 + \frac{\delta m^2}{2} \phi^2 \quad . \quad (4.2)$$

The renormalization of ϕ^4 theory in 1+1 dimension is well known [13] and can be achieved by introducing a mass counterterm δm^2 given as

$$\delta m^2 = - \frac{g}{4L} \sum_k \frac{1}{\sqrt{k^2 + m^2}} \quad . \quad (4.3)$$

The expansion of the hamiltonian on the basis of natural orbitals follows directly from the discussion of section II. Therefore we have now all the necessary ingredients to implement the proposed approximation to the collisional dynamics. This is a lengthly but straightforward algebraic exercise. The resulting equations of motion are

$$\begin{aligned} i \dot{A}_k &= \frac{k_0}{2} (A_k + A_{-k}^* + \frac{m^2 + k^2}{2k_0} (A_k - A_{-k}^*) + \frac{g}{24L} \sum_{k_1 k_2 k_3} \frac{1}{\sqrt{k_{01} k_{02} k_{03} k_0}} \\ &\quad \left\{ \left(\delta_{k_1+k_2+k_3+k, 0} A_{k_1}^* A_{k_2}^* A_{k_3}^* + \delta_{k_1+k_2+k_3-k, 0} A_{k_1} A_{k_2} A_{k_3} \right) \right. \\ &\quad \left. 3 \left(\delta_{k_1+k_2-k_3+k, 0} A_{k_1}^* A_{k_2}^* A_{k_3} + \delta_{k_1+k_2-k_3-k, 0} A_{k_1} A_{k_2} A_{k_3}^* \right) \right\} \\ &+ \frac{g}{8L} \sum_{k_1 k_2 k_3} \frac{1}{\sqrt{k_{01} k_{02} k_{03} k_0}} \\ &\quad \left(\delta_{k_1+k_2-k_3, 0} A_{k_1}^* + \delta_{k_1-k-k_2+k_3, 0} A_{k_1} \right) \sum_{\alpha} T_{k_2 \alpha} T_{k_3 \alpha}^* (1 + 2p_{\alpha}) \\ &- \frac{g}{8Lk_0} (A_k + A_{-k}^* \sum_{k'} \frac{1}{\sqrt{k'^2 + m^2}} + i \Gamma_A(t) \quad , \\ \dot{p}_{\alpha} &= \Gamma_p(t) \quad , \end{aligned} \quad (4.4)$$

$$\begin{aligned} &(1 + p_{\alpha} + p_{\beta}) g_{\alpha \beta} \\ &= \sum_k \left[- \frac{k_0}{2} (U_{k\alpha}^* + V_{-k\alpha}^*)(U_{-k\beta}^* + V_{k\beta}^*) + \frac{k^2 + m^2}{2k_0} (U_{k\alpha}^* - V_{-k\alpha}^*)(U_{-k\beta}^* - V_{k\beta}^*) \right] \\ &+ \frac{g}{8L} \sum_{k_1 k_2 k_3 k_4} \frac{1 + p_{\alpha} + p_{\beta}}{\sqrt{k_{01} k_{02} k_{03} k_{04}}} \left\{ \left(\delta_{k_1+k_2+k_3+k_4, 0} A_{k_1}^* A_{k_2}^* + \delta_{k_1+k_2-k_3-k_4, 0} A_{k_1} A_{k_2} \right) \right. \end{aligned}$$

$$\begin{aligned}
& + 2\delta_{k_1-k_2+k_3+k_4,0} A_{k_1}^* A_{k_2} T_{k_3\alpha}^* T_{k_4\beta} \} \\
& + \frac{g}{8L} \sum_{k_1 k_2 k_3 k_4} \delta_{k_1+k_2+k_3-k_4,0} \frac{1+p_\alpha+p_\beta}{\sqrt{k_{01} k_{02} k_{03} k_{04}}} T_{k_1\alpha}^* T_{k_2\beta}^* \sum_{\alpha'} T_{k_3\alpha'}^* T_{k_4\alpha'} (1+2p_{\alpha'}) \\
& - \frac{g}{8L} \sum_{k_1} \frac{1+p_\alpha+p_\beta}{k_{01}} T_{k_1\alpha}^* T_{k_2\beta}^* \sum_{k_2} \frac{1}{\sqrt{k_2^2+m^2}} - i\Gamma_g(t) \quad . \tag{4.5}
\end{aligned}$$

$$\begin{aligned}
& (p_\beta - p_\alpha) h_{\alpha\beta} \\
& = \sum_k \left[\frac{k_0}{2} (U_{k\alpha}^* + V_{-k\alpha}^*)(U_{k\beta} + V_{-k\beta}) + \frac{k^2+m^2}{2k_0} T_{k\alpha}^* T_{k\beta} \right] (p_\beta - p_\alpha) \\
& + \frac{g}{16L} \sum_{k_1 k_2 k_3 k_4} \frac{1}{\sqrt{k_{01} k_{02} k_{03} k_{04}}} \\
& \quad \left\{ \delta_{k_1+k_2+k_3-k_4,0} (A_{k_1}^* A_{k_2}^* T_{k_3\alpha}^* T_{k_4\beta} + A_{k_1} A_{k_2} T_{k_3\beta} T_{k_4\alpha}^*) \right. \\
& + \delta_{k_1+k_2-k_3+k_4,0} (A_{k_1} A_{k_2} T_{k_3\alpha}^* T_{k_4\beta} + A_{k_1}^* A_{k_2}^* T_{k_3\beta} T_{k_4\alpha}^*) \\
& + 2\delta_{k_1-k_2+k_3-k_4,0} (A_{k_1}^* A_{k_2} T_{k_3\alpha}^* T_{k_4\beta} + A_{k_1} A_{k_2}^* T_{k_3\beta} T_{k_4\alpha}^*) \left. \right\} (p_\beta - p_\alpha) \\
& + \frac{g}{8L} \sum_{k_1 k_2 k_3 k_4} \frac{\delta_{k_1+k_2-k_3-k_4,0}}{\sqrt{k_{01} k_{02} k_{03} k_{04}}} T_{k_1\alpha}^* T_{k_3\beta} \sum_{\alpha'} T_{k_2\alpha'}^* T_{k_4\alpha'} (p_\beta - p_\alpha) \\
& - \frac{g}{8L} \sum_{k_1} \frac{1}{k_{01}} T_{k_1\alpha}^* T_{k_1\beta} \sum_{k_2} \frac{1}{\sqrt{k_2^2+m^2}} (p_\beta - p_\alpha) + i\Gamma_h(t) \quad , \tag{4.6}
\end{aligned}$$

where

$$T_{k\alpha} = U_{k\alpha} - V_{k\alpha}$$

and the collision integrals $\Gamma(t)$ are

$$\begin{aligned}
\Gamma_A(t) &= \frac{g^2}{96L^2} \sum_{k_1 k_2 k_3} \frac{1}{\sqrt{k_{01} k_{02} k_{03} k_0}} \sum_{\alpha_1 \alpha_2 \alpha_3} \\
& \quad \left\{ \delta_{k_1+k_2+k_3+k,0} T_{k_1\alpha_1}^* T_{k_2\alpha_2}^* T_{k_3\alpha_3}^* I_{\alpha_1\alpha_2\alpha_3}^{(4)} \right. \\
& - \delta_{k_1+k_2+k_3-k,0} T_{k_1\alpha_1} T_{k_2\alpha_2} T_{k_3\alpha_3} I_{\alpha_1\alpha_2\alpha_3}^{(4)*} \\
& + 3\delta_{k_1+k_2+k-k_3,0} T_{k_1\alpha_1}^* T_{k_2\alpha_2}^* T_{k_3\alpha_3} I_{\alpha_1\alpha_2\alpha_3}^{(5)} \\
& - 3\delta_{k_1+k_2-k-k_3,0} T_{k_1\alpha_1} T_{k_2\alpha_2} T_{k_3\alpha_3}^* I_{\alpha_1\alpha_2\alpha_3}^{(5)*} \left. \right\} \quad , \tag{4.7}
\end{aligned}$$

$$\begin{aligned}
\Gamma_P(t) &= \frac{g^2}{96L^2} \sum_{k_1 k_2 k_3 k_4} \frac{1}{\sqrt{k_{01} k_{02} k_{03} k_{04}}} \sum_{\alpha_1 \alpha_2 \alpha_3} \\
& \quad \left\{ \delta_{k_1+k_2+k_3+k_4,0} T_{k_1\alpha_1}^* T_{k_2\alpha_2}^* T_{k_3\alpha_3}^* T_{k_4\alpha}^* I_{\alpha_1\alpha_2\alpha_3\alpha}^{(1)} \right.
\end{aligned}$$

$$\begin{aligned}
& + 3\delta_{k_1+k_2+k_3-k_4,0} T_{k_1\alpha_1}^* T_{k_2\alpha_2}^* T_{k_3\alpha}^* T_{k_4\alpha_3} I_{\alpha_1\alpha_2\alpha_3}^{(2)} \\
& - \delta_{k_1+k_2+k_3-k_4,0} T_{k_1\alpha_1}^* T_{k_2\alpha_2}^* T_{k_3\alpha_3}^* T_{k_4\alpha} I_{\alpha_1\alpha_2\alpha_3\alpha}^{(2)} \\
& + 3\delta_{k_1+k_2-k_3-k_4,0} T_{k_1\alpha}^* T_{k_2\alpha_1}^* T_{k_3\alpha_2}^* T_{k_4\alpha_3} I_{\alpha\alpha_1\alpha_2\alpha_3}^{(3)} \Big\} \\
& + \frac{g^2}{32L^2} \sum_{k_1k_2k_3k_4} \frac{1}{\sqrt{k_{01}k_{02}k_{03}k_{04}}} \sum_{\alpha_1\alpha_2} \\
& \quad \left\{ \left(\delta_{k_1+k_2+k_3+k_4,0} A_{k_1} + \delta_{-k_1+k_2+k_3+k_4,0} A_{k_1}^* \right) T_{k_2\alpha_1} T_{k_3\alpha_2} T_{k_4\alpha} I_{\alpha_1\alpha_2\alpha}^{(4)} \right. \\
& \quad 2 \left(\delta_{k_1-k_2-k_3+k_4,0} A_{k_1} + \delta_{k_1+k_2+k_3-k_4,0} A_{k_1}^* \right) T_{k_2\alpha_1}^* T_{k_3\alpha}^* T_{k_4\alpha_2} I_{\alpha_1\alpha\alpha_2}^{(5)} \\
& \quad \left. - \left(\delta_{k_1-k_2-k_3+k_4,0} A_{k_1} + \delta_{k_1+k_2+k_3-k_4,0} A_{k_1}^* \right) T_{k_2\alpha_1}^* T_{k_3\alpha_2}^* T_{k_4\alpha} I_{\alpha_1\alpha_2\alpha}^{(5)} \right\} \\
& + \text{c. c.} \quad , \tag{4.8}
\end{aligned}$$

$$\begin{aligned}
\Gamma_h(t) &= \frac{g^2}{96L^2} \sum_{k_1k_2k_3k_4} \frac{1}{\sqrt{k_{01}k_{02}k_{03}k_{04}}} \sum_{\alpha_1\alpha_2\alpha_3} \\
& \quad \left\{ \delta_{k_1+k_2+k_3+k_4,0} T_{k_1\alpha_1}^* T_{k_2\alpha_2}^* T_{k_3\alpha_3}^* T_{k_4\alpha}^* I_{\alpha_1\alpha_2\alpha_3\beta}^{(1)} \right. \\
& + 3\delta_{k_1+k_2+k_3-k_4,0} T_{k_1\alpha_1}^* T_{k_2\alpha_2}^* T_{k_3\alpha}^* T_{k_4\alpha_3} I_{\alpha_1\alpha_2\beta\alpha_3}^{(2)} \\
& - \delta_{k_1+k_2+k_3-k_4,0} T_{k_1\alpha_1} T_{k_2\alpha_2} T_{k_3\alpha_3} T_{k_4\alpha}^* I_{\alpha_1\alpha_2\alpha_3\beta}^{(2)*} \\
& + 3\delta_{k_1+k_2-k_3-k_4,0} T_{k_1\alpha}^* T_{k_2\alpha_1}^* T_{k_3\alpha_2}^* T_{k_4\alpha_3} I_{\beta\alpha_1\alpha_2\alpha_3}^{(3)} \Big\} \\
& + \frac{g^2}{32L^2} \sum_{k_1k_2k_3k_4} \frac{1}{\sqrt{k_{01}k_{02}k_{03}k_{04}}} \sum_{\alpha_1\alpha_2} \\
& \quad \left\{ \left(\delta_{-k_1+k_2+k_3+k_4,0} A_{k_1} + \delta_{k_1+k_2+k_3+k_4,0} A_{k_1}^* \right) T_{k_2\alpha_1}^* T_{k_3\alpha_2}^* T_{k_4\alpha}^* I_{\alpha_1\alpha_2\beta}^{(4)} \right. \\
& \quad 2 \left(\delta_{k_1-k_2-k_3+k_4,0} A_{k_1} + \delta_{k_1+k_2+k_3-k_4,0} A_{k_1}^* \right) T_{k_2\alpha_1}^* T_{k_3\alpha}^* T_{k_4\alpha_2} I_{\alpha_1\beta\alpha_2}^{(5)} \\
& \quad \left. - \left(\delta_{k_1-k_2-k_3+k_4,0} A_{k_1} + \delta_{k_1+k_2+k_3-k_4,0} A_{k_1}^* \right) T_{k_2\alpha_1} T_{k_3\alpha_2} T_{k_4\alpha}^* I_{\alpha_1\alpha_2\beta}^{(5)} \right\} \\
& + \text{c. c.} \quad (\alpha \leftrightarrow \beta) \quad , \tag{4.9}
\end{aligned}$$

$$\begin{aligned}
\Gamma_g(t) &= \frac{g^2}{96L^2} \sum_{k_1k_2k_3k_4} \frac{1}{\sqrt{k_{01}k_{02}k_{03}k_{04}}} \sum_{\alpha_1\alpha_2\alpha_3} \\
& \quad \left\{ \delta_{k_1+k_2+k_3-k_4,0} T_{k_1\alpha_1} T_{k_2\alpha_2} T_{k_3\alpha_3} T_{k_4\alpha}^* I_{\alpha_1\alpha_2\alpha_3\beta}^{(1)*} \right. \\
& + 3\delta_{k_1+k_2-k_3-k_4,0} T_{k_1\alpha_1} T_{k_2\alpha_2} T_{k_3\alpha}^* T_{k_4\alpha_3}^* I_{\alpha_1\alpha_2\beta\alpha_3}^{(2)*} \\
& - \delta_{k_1+k_2+k_3+k_4,0} T_{k_1\alpha_1}^* T_{k_2\alpha_2}^* T_{k_3\alpha_3}^* T_{k_4\alpha}^* I_{\alpha_1\alpha_2\alpha_3\beta}^{(2)} \\
& + 3\delta_{k_1+k_2+k_3-k_4,0} T_{k_1\alpha_1}^* T_{k_2\alpha_2}^* T_{k_3\alpha}^* T_{k_4\alpha_3} I_{\alpha_1\alpha_2\beta\alpha_3}^{(3)} \Big\} \\
& + \frac{g^2}{32L^2} \sum_{k_1k_2k_3k_4} \frac{1}{\sqrt{k_{01}k_{02}k_{03}k_{04}}} \sum_{\alpha_1\alpha_2}
\end{aligned}$$

$$\begin{aligned}
& \left\{ \left(\delta_{k_1+k_2+k_3-k_4,0} A_{k_1} + \delta_{k_1-k_2-k_3+k_4,0} A_{k_1}^* \right) T_{k_2\alpha_1} T_{k_3\alpha_2} T_{k_4\alpha}^* I_{\alpha_1\alpha_2\beta}^{(4)*} \right. \\
& 2 \left(\delta_{k_1-k_2-k_3+k_4,0} A_{k_1} + \delta_{k_1+k_2+k_3-k_4,0} A_{k_1}^* \right) T_{k_2\alpha_1} T_{k_3\alpha}^* T_{k_4\alpha_2}^* I_{\alpha_1\beta\alpha_2}^{(5)*} \\
& - \left(\delta_{-k_1+k_2+k_3+k_4,0} A_{k_1} + \delta_{k_1+k_2+k_3+k_4,0} A_{k_1}^* \right) T_{k_2\alpha_1}^* T_{k_3\alpha_2}^* T_{k_4\alpha}^* I_{\alpha_1\alpha_2\beta}^{(5)} \left. \right\} \\
& + \alpha \leftrightarrow \beta \quad .
\end{aligned} \tag{4.10}$$

In these equations we used the abbreviations

$$\begin{aligned}
I_{\alpha_1\alpha_2\alpha_3\alpha_4}^{(1)}(t) &= \int_0^t dt' \sum_{k'_1 k'_2 k'_3 k'_4} \frac{\delta_{k'_1+k'_2+k'_3+k'_4,0}}{\sqrt{k'_{01} k'_{02} k'_{03} k'_{04}}} \sum_{\gamma_1 \gamma_2 \gamma_3 \gamma_4} \left(T_{k'_1 \gamma_1} T_{k'_2 \gamma_2} T_{k'_3 \gamma_3} T_{k'_4 \gamma_4} \right)_{t'} \\
& \left(1 + \sum_i^4 p_{\gamma_i} + \sum_{i<j}^4 p_{\gamma_i} p_{\gamma_j} + \sum_{i<j<\ell}^4 p_{\gamma_i} p_{\gamma_j} p_{\gamma_\ell} \right)_{t'} \\
& \left(M_{\alpha_1 \gamma_1}^*(t, t') M_{\alpha_2 \gamma_2}^*(t, t') M_{\alpha_3 \gamma_3}^*(t, t') M_{\alpha_4 \gamma_4}^*(t, t') \right) \quad ,
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
I_{\alpha_1\alpha_2\alpha_3\alpha_4}^{(2)}(t) &= \int_0^t dt' \sum_{k'_1 k'_2 k'_3 k'_4} \frac{\delta_{k'_1+k'_2+k'_3-k'_4,0}}{\sqrt{k'_{01} k'_{02} k'_{03} k'_{04}}} \sum_{\gamma_1 \gamma_2 \gamma_3 \gamma_4} \left(T_{k'_1 \gamma_1} T_{k'_2 \gamma_2} T_{k'_3 \gamma_3} T_{k'_4 \gamma_4}^* \right)_{t'} \\
& \left(p_{\gamma_4} - p_{\gamma_1} p_{\gamma_2} p_{\gamma_3} + p_{\gamma_4} \sum_i^3 p_{\gamma_i} + p_{\gamma_4} \sum_{i<j}^3 p_{\gamma_i} p_{\gamma_j} \right)_{t'} \\
& \left(M_{\alpha_1 \gamma_1}^*(t, t') M_{\alpha_2 \gamma_2}^*(t, t') M_{\alpha_3 \gamma_3}^*(t, t') M_{\alpha_4 \gamma_4}(t, t') \right) \quad ,
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
I_{\alpha_1\alpha_2\alpha_3\alpha_4}^{(3)}(t) &= \int_0^t dt' \sum_{k'_1 k'_2 k'_3 k'_4} \frac{\delta_{k'_1+k'_2-k'_3-k'_4,0}}{\sqrt{k'_{01} k'_{02} k'_{03} k'_{04}}} \sum_{\gamma_1 \gamma_2 \gamma_3 \gamma_4} \left(T_{k'_1 \gamma_1} T_{k'_2 \gamma_2} T_{k'_3 \gamma_3}^* T_{k'_4 \gamma_4}^* \right)_{t'} \\
& [p_{\gamma_3} p_{\gamma_4} (1 + p_{\gamma_1}) (1 + p_{\gamma_2}) - p_{\gamma_1} p_{\gamma_2} (1 + p_{\gamma_3}) (1 + p_{\gamma_4})]_{t'} \\
& \left(M_{\alpha_1 \gamma_1}^*(t, t') M_{\alpha_2 \gamma_2}^*(t, t') M_{\alpha_3 \gamma_3}(t, t') M_{\alpha_4 \gamma_4}(t, t') \right) \quad ,
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
I_{\alpha_1\alpha_2\alpha_3}^{(4)}(t) &= \int_0^t dt' \sum_{k'_1 k'_2 k'_3 k'_4} \frac{1}{\sqrt{k'_{01} k'_{02} k'_{03} k'_{04}}} \sum_{\gamma_1 \gamma_2 \gamma_3} \left(T_{k'_2 \gamma_1} T_{k'_3 \gamma_2} T_{k'_4 \gamma_3} \right)_{t'} \\
& \left(\delta_{k'_1+k'_2+k'_3+k'_4,0} A_{k'_1} + \delta_{-k'_1+k'_2+k'_3+k'_4,0} A_{k'_1}^* \right)_{t'} \\
& \left(1 + \sum_i^3 p_{\gamma_i} + \sum_{i<j}^3 p_{\gamma_i} p_{\gamma_j} \right)_{t'} \\
& \left(M_{\alpha_1 \gamma_1}^*(t, t') M_{\alpha_2 \gamma_2}^*(t, t') M_{\alpha_3 \gamma_3}^*(t, t') \right) \quad ,
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
I_{\alpha_1\alpha_2\alpha_3}^{(5)}(t) &= \int_0^t dt' \sum_{k'_1 k'_2 k'_3 k'_4} \frac{1}{\sqrt{k'_{01} k'_{02} k'_{03} k'_{04}}} \sum_{\gamma_1 \gamma_2 \gamma_3} \left(T_{k'_2 \gamma_1} T_{k'_3 \gamma_2} T_{k'_4 \gamma_3}^* \right)_{t'} \\
&\quad \left(\delta_{k'_1 + k'_2 + k'_3 - k'_4, 0} A_{k'_1} + \delta_{k'_1 - k'_2 - k'_3 + k'_4, 0} A_{k'_1}^* \right)_{t'} \\
&\quad [p_{\gamma_3} (1 + p_{\gamma_1} + p_{\gamma_2}) - p_{\gamma_1} p_{\gamma_2}]_{t'} \\
&\quad \left(M_{\alpha_1 \gamma_1}^*(t, t') M_{\alpha_2 \gamma_2}^*(t, t') M_{\alpha_3 \gamma_3}(t, t') \right)_{t'} .
\end{aligned} \tag{4.15}$$

In summary, we have presented in this paper a framework to study real-time evolution of scalar field theory. The technique has been applied to nuclear many-body theory and extended recently in the context of homogeneous field configurations. Here we discuss this general problem when the spatial dependence is important. Here we show that the spatial dependence can be treated in the general orbital representation and the dynamics of these orbitals are expressed in a closed form by a set of selfconsistent equations. In this way, the time-dependent technique can be used to improve the usual gaussian-like mean field approximation, where the collisional dynamics are given by appropriate memorial integrals. We have illustrated these procedures within the simplest context of self-interacting ϕ^4 theory.

Appendix A: Projection Technique in Field Theory

In order to calculate equations of motion (2.24), (2.32), (2.33) and (2.34) we first decompose F in two parts

$$F = F_0(t) + F'(t) \tag{A.1}$$

where $F_0(t)$ is the exponential of a one-boson density given in (3.39). A crucial point is to observe that $F_0(t)$ can be seen as a time-dependent projection of F , i.e., For the explicit construction of $\mathbb{P}(t)$ we require, in addition to eqs. (3.3), the condition

$$i \dot{F}_0(t) = [F_0(t), H] + \mathbb{P}(t)[H, F] \tag{A.2}$$

which is the Heisenberg picture counterpart of the Schrödinger picture condition used in ref[11] to determine \mathbb{P} uniquely. The resulting form for $\mathbb{P}(t)$ is (see ref.[6] for details of the derivation)

$$\mathbb{P} \cdot = \left\{ \left[1 - \sum_{\alpha} \frac{d_{\alpha}^{\dagger} d_{\alpha} - p_{\alpha}}{1 + p_{\alpha}} \right] \text{Tr}(\cdot) + \sum_{\alpha_1 \alpha_2} \frac{d_{\alpha_1}^{\dagger} d_{\alpha_2} - p_{\alpha_2} \delta_{\alpha_1 \alpha_2}}{p_{\alpha_2} (1 + p_{\alpha_1})} \text{Tr} \left(d_{\alpha_2}^{\dagger} d_{\alpha_1} \cdot \right) \right.$$

$$\begin{aligned}
& + \sum_{\alpha} \left[\frac{d_{\alpha}}{p_{\alpha}} \text{Tr}(d_{\alpha}^{\dagger} \cdot) + \frac{d_{\alpha}^{\dagger}}{1+p_{\alpha}} \text{Tr}(d_{\alpha} \cdot) \right] + \sum_{\alpha_1 \alpha_2} \left[\frac{d_{\alpha_1} d_{\alpha_2}}{2p_{\alpha_1} p_{\alpha_2}} \text{Tr}(d_{\alpha_2}^{\dagger} d_{\alpha_1}^{\dagger} \cdot) \right. \\
& \left. + \frac{d_{\alpha_1}^{\dagger} d_{\alpha_2}^{\dagger}}{2(1+p_{\alpha_1})(1+p_{\alpha_2})} \text{Tr}(d_{\alpha_2} d_{\alpha_1} \cdot) \right] \Big\} F_0 \quad . \tag{A.3}
\end{aligned}$$

The next step is to obtain a differential equation of $F'(t)$. This follows immediately from eqs. (3.1) and (3.4),

$$\left(i \frac{d}{dt} - \mathbb{P}(t) \mathbb{L} \right) F'(t) = \mathbb{Q}(t) \mathbb{L} F_0(t) \quad , \tag{A.4}$$

where we introduced the operators

$$\mathbb{Q}(t) = \mathbb{L} - \mathbb{P}(t) \quad , \quad \mathbb{L} \cdot = [H, \cdot] \quad . \tag{A.5}$$

This equation has the formal solution

$$F'(t) = \mathbb{G}(t, 0) F'(0) - i \int_0^t dt' \mathbb{G}(t, t') \mathbb{Q}(t') \mathbb{L} F_0(t') \quad , \tag{A.6}$$

where $\mathbb{G}(t, t')$ is the time-ordered Green's Function

$$\mathbb{G}(t, t') = T \exp i \int_{t'}^t d\tau \mathbb{P}(\tau) \mathbb{L} \quad . \tag{A.7}$$

A systematic expansion to treat the memory integral of the equation (3.8) has been discussed in ref.[11] in the Schrödinger picture. The implementation of the corresponding expansion in the Heisenberg picture consists in approximating the time evolution of the field operators by the simpler mean-field Hamiltonian

$$\begin{aligned}
H_0 &= P^{\dagger} H + \sum_{\alpha} d_{\alpha}^{\dagger} \text{Tr}[d_{\alpha}, H] F' - \sum_{\alpha} d_{\alpha} \text{Tr}[d_{\alpha}^{\dagger}, H] F' \\
&+ \sum_{\alpha_1 \alpha_2} \frac{d_{\alpha_1}^{\dagger} d_{\alpha_2}^{\dagger}}{2(1+p_{\alpha_1}+p_{\alpha_2})} \text{Tr}[d_{\alpha_1} d_{\alpha_2}, H] F' \\
&- \sum_{\alpha_1 \alpha_2} \frac{d_{\alpha_1} d_{\alpha_2}}{2(1+p_{\alpha_1}+p_{\alpha_2})} \text{Tr}[d_{\alpha_1}^{\dagger} d_{\alpha_2}^{\dagger}, H] F' \quad . \tag{A.8}
\end{aligned}$$

Using this approximation, one can solve explicitly the Heisenberg field operator equation as

$$d_\alpha(t) \simeq \sum_{\gamma} M_{\alpha\gamma}(t, t') d_\gamma(t') \quad , \quad (\text{A.9})$$

where the matrix $M_{\alpha\gamma}$ is the solution of the matrix equation

$$i \dot{M}(t, t') = F(t) M(t, t') \quad , \quad (\text{A.10})$$

and the matrix F involves matrix h and mean-field energy (see appendix A for details of the derivation).

In this way, the authors of the ref.[11] devised a systematic expansion for the correlation density $F'(t)$. The corresponding expansion in the Heisenberg picture is

$$\begin{aligned} F'(t) &= \mathbb{G}(t, 0) F'(0) - i \int_0^t dt'_1 Q(t_1) L F_0(t_1) \\ &- \int_0^t dt_1 \left[\int_{t_1}^t dt_2 Q(t_2) (L - L_0(t_2)) \right] Q(t_1) L F_0(t_1) + \dots \quad , \end{aligned} \quad (\text{A.11})$$

where $L_0 \cdot = [H_0, \cdot]$. In what follows we restrict ourselves to initial conditions such that $F'(0) = 0$ and to the lowest approximation for $F'(t)$. Therefore, the evaluation of the equations of motion involves traces of the type

$$\text{Tr} [\hat{O}(t), H] F_0(t) - i \text{Tr} [\hat{O}(t), H] \int_0^t dt' Q(t') [H, F_0(t')] \quad , \quad (\text{A.12})$$

where $\hat{O}(t)$ can be $d_\alpha(t)$, $d_\alpha^\dagger(t) d_\beta(t)$, $d_\alpha(t) d_\beta(t)$, and operators at different times are related by equation (3.11).

Appendix B: Approximation for the Time Evolution of the Heisenberg Field Operator

In section II, we have discussed that our approximation consists in replacing the time evolution of the field operator by a simpler mean-field Hamiltonian given by equation (3.10). We show now that this allows one to solve the Heisenberg operator equation

$$i \dot{d}_\alpha = [d_\alpha, H_0] - \sum_{\gamma} h_{\alpha\gamma} d_\alpha - \sum_{\gamma} g_{\gamma\alpha} d_\gamma^\dagger - \text{Tr} [d_\alpha, H] \quad . \quad (\text{B.1})$$

The last three terms are the explicit time dependence of $d_\alpha(t)$ related to shifted amplitudes $A_k(t)$ and to effects of the general Bogoliubov transformation for operators (2.19).

First, we write $P^\dagger H$, using cyclic properties of the traces, as (see ref. [] for the construction of P^\dagger)

$$\begin{aligned}
P^\dagger H &= \left(1 - \sum_\alpha \frac{d_\alpha^\dagger d_\alpha - p_\alpha}{1 + p_\alpha}\right) \text{Tr}(H F_0) + \sum_{\alpha_1 \alpha_2} \frac{d_{\alpha_1}^\dagger d_{\alpha_2} - p_{\alpha_1} \delta_{\alpha_1 \alpha_2}}{P_{\alpha_2}(1 + p_{\alpha_1})} \text{Tr}(d_{\alpha_2}^\dagger d_{\alpha_1} H F_0) \\
&- \sum_\alpha d_\alpha \text{Tr}[d_\alpha^\dagger, H] F_0 + \sum_\alpha d_\alpha^\dagger \text{Tr}[d_\alpha, H] F_0 \\
&- \sum_{\alpha_1 \alpha_2} \frac{d_{\alpha_1} d_{\alpha_2}}{2(1 + p_{\alpha_1} + p_{\alpha_2})} \text{Tr}[d_{\alpha_2}^\dagger d_{\alpha_1}^\dagger, H] F_0 \\
&+ \sum_{\alpha_1 \alpha_2} \frac{d_{\alpha_1}^\dagger d_{\alpha_2}^\dagger}{2(1 + p_{\alpha_1} + p_{\alpha_2})} \text{Tr}[d_{\alpha_2} d_{\alpha_1}, H] F_0
\end{aligned} \tag{B.2}$$

Hence, equation (3.10) becomes

$$\begin{aligned}
H_0 &= \left(1 - \sum_\alpha \frac{d_\alpha^\dagger d_\alpha - p_\alpha}{1 + p_\alpha}\right) \text{Tr}(H F_0) + \sum_{\alpha_1 \alpha_2} \frac{d_{\alpha_1}^\dagger d_{\alpha_2} - p_{\alpha_1} \delta_{\alpha_1 \alpha_2}}{p_{\alpha_2}(1 + p_{\alpha_1})} \text{Tr}(d_{\alpha_2}^\dagger d_{\alpha_1} H F_0) \\
&- \sum_\alpha d_\alpha \text{Tr}[d_\alpha^\dagger, H] F + \sum_\alpha d_\alpha^\dagger \text{Tr}[d_\alpha, H] F \\
&- \sum_{\alpha_1 \alpha_2} \frac{d_{\alpha_1} d_{\alpha_2}}{2(1 + p_{\alpha_1} + p_{\alpha_2})} \text{Tr}[d_{\alpha_2}^\dagger d_{\alpha_1}^\dagger, H] F \\
&+ \sum_{\alpha_1 \alpha_2} \frac{d_{\alpha_1}^\dagger d_{\alpha_2}^\dagger}{2(1 + p_{\alpha_1} + p_{\alpha_2})} \text{Tr}[d_{\alpha_2} d_{\alpha_1}, H] F
\end{aligned} \tag{B.3}$$

Using (A.3) in (A.1) one obtains immediately

$$i \dot{d}_\alpha = - \frac{\text{Tr}(H F_0)}{1 + p_\alpha} d_\alpha + \sum_{\alpha'} \frac{d_{\alpha'}}{p_{\alpha'}(1 + p_\alpha)} \text{Tr}(d_{\alpha'}^\dagger d_\alpha H F_0) - \sum_\alpha h_{\alpha\gamma} d_\gamma \quad . \tag{B.4}$$

The calculation for the first two terms is straightforward. It yields

$$\frac{1}{2} \sum_{\alpha'} \sum_k \left[k_0 (U_{k\alpha}^* + V_{-k\alpha}^*)(U_{k\alpha'} + V_{-k\alpha'}) + \frac{k^2 + m^2}{k_0} T_{k\alpha}^* T_{k\alpha'} \right] d_{\alpha'}$$

$$\begin{aligned}
& + \frac{g}{16L} \sum_{k_1 k_2 k_3 k_4} \frac{1}{\sqrt{k_{01} k_{02} k_{03} k_{04}}} \sum_{\alpha'} \\
& \times \left\{ \delta_{k_1+k_2-k_3-k_4, 0} \left(A_{k_1} A_{k_2} T_{k_3 \alpha}^* T_{k_4 \alpha'} + A_{k_1}^* A_{k_2}^* T_{k_3 \alpha'} T_{k_4 \alpha} \right) \right. \\
& + \delta_{k_1+k_2+k_3-k_4, 0} \left(A_{k_1} A_{k_2} T_{k_3 \alpha'} T_{k_4 \alpha}^* + A_{k_1}^* A_{k_2}^* T_{k_3 \alpha}^* T_{k_4 \alpha'} \right) \\
& + 2\delta_{k_1-k_2+k_3-k_4, 0} \left(A_{k_1}^* A_{k_2} T_{k_3 \alpha}^* T_{k_4 \alpha'} + A_{k_1} A_{k_2}^* T_{k_3 \alpha'} T_{k_4 \alpha}^* \right) \left. \right\} d_{\alpha'} \\
& + \frac{g}{4L} \sum_{k_1 k_2 k_3 k_4} \frac{\delta_{k_1+k_2-k_3-k_4, 0}}{\sqrt{k_{01} k_{02} k_{03} k_{04}}} \sum_{\alpha_1 \alpha_2} T_{k_1 \alpha}^* T_{k_2 \alpha_2}^* T_{k_3 \alpha_1} T_{k_4 \alpha_2} p_{\alpha_2} d_{\alpha_1} \\
& + \frac{g}{8L} \sum_{k_1 k_2 k_3 k_4} \frac{\delta_{k_1+k_2-k_3-k_4, 0}}{\sqrt{k_{01} k_{02} k_{03} k_{04}}} \sum_{\alpha_1 \alpha_2} T_{k_1 \alpha_1}^* T_{k_2 \alpha_2}^* T_{k_3 \alpha_2} T_{k_4 \alpha_1} d_{\alpha_1} d_{\alpha_1} \\
& - \frac{g}{8L} \sum_{k_1 k_2} \frac{1}{\sqrt{k_1^2 + m^2}} \frac{1}{k_{02}} \sum_{\alpha_1} T_{k_2 \alpha}^* T_{k_2 \alpha_1} d_{\alpha_1} \\
& \equiv \sum_{\gamma} B_{\alpha \gamma} d_{\gamma} \quad .
\end{aligned} \tag{B.5}$$

One finds finally the Heisenberg operator equation as being

$$\begin{aligned}
i \dot{d}_{\alpha} &= \sum_{\gamma} B_{\alpha \gamma} d_{\gamma} - \sum_{\gamma} h_{\alpha \gamma} d_{\gamma} \\
&\equiv \sum_{\gamma} F_{\alpha \gamma} d_{\gamma}
\end{aligned} \tag{B.6}$$

or in the matrix form

$$i \dot{d}(t) = F(t) d(t) \quad . \tag{B.7}$$

It is easy to see that the solution for (A.7) is

$$d(t) = M(t, t') d(t') \quad , \tag{B.8}$$

where $M(t, t')$ is the solution of the matrix equation

$$i \dot{M}(t, t') = F(t) M(t, t') \tag{B.9}$$

with initial condition

$$M(t, t') = 1 \quad . \quad (\text{B.10})$$

References

- [1] M.B. Hindmarsh and T.W.Kibble, *Rep. Prog. Phys.***58**, 477 (1995).
- [2] D. Boyanovsky, M. D'attanasio, H.J de Vega and R. Holman, *Phys. Rev.* **D54**, 1748 (1996); Sean Gavin, *Nucl. Phys.* **A 590**, 163c (1995).
- [3] H.T.C. Stoof, *Phys. Rev.* **A 45**, 8398 (1992); P.A. Ruprecht, M.J. Holland and K. Burnett, *ibid.* **A 51**, 4704; K.N. Ilinski and A.S. Stepanenko: *Hydrodynamics of a Bose condensate: beyond the mean field approximation*, preprint cond-mat/9607202; H.T.C. Stoof: *Initial stage of Bose-Einstein Condensation*, preprint cond-mat/9608151.
- [4] See, e.g. R. Jackiw, *Physica A* **158**, 269 (1989).
- [5] A. F. R. de Toledo Piza, in *Time-Dependent Hartree-Fock and Beyond*, edited by K. Goeke and P.-G. Reinhardt, Lectures Notes in Physics 171 (Springer-Verlag, Berlin, 1982); M. C. Nemes and A. F. R. de Toledo Piza, *Phys. Rev.* **C 27**, 862 (1983); B. V. Carlson, M. C. Nemes and A. F. R. de Toledo Piza, *Nucl. Phys* **A 457**, 261 (1986); M. C. Nemes and A. F. R. de Toledo Piza, *Physica A* **137**, 367 (1986).
- [6] Chi-Yong Lin and A. F. R. de Toledo Piza, *Phys. Rev.* **D 46**, 742 (1992).
- [7] C-Y. Lin and A. F. R. de Toledo Piza, *Mod. Phys. Lett.* **A 5**, 1605 (1990);
- [8] P.M. Stevenson, *Phys. Rev.* **D 30**, 1712 (1984).
- [9] P. Ring and P. Schuck, “*The Nuclear Many-Body Problem*”, Springer-Verlag (1980).
- [10] C. R. Willis and R. H. Picard, *Phys. Rev.* **A9**, 1343 (1974).
- [11] P. Buck, H. Feldmeier and M.C. Nemes, *Ann. Phys. (N.Y.)* **185**, 170 (1988)
- [12] P. L. Natti and A. F. R. de Toledo Piza, *Phys. Rev.* **D 54**, 7867 (1996); and P. L. Natti and A. F. R. de Toledo Piza, *Phys. Rev.* **D 55**, 3403 (1997).
- [13] P.M. Stevenson, *Phys. Rev.* **D 32**, 1398 (1985).